CS155: Modeling

Curves
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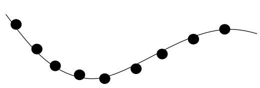
Overview

- Curves
  - interpolating curves
  - hermitian splines
  - bezier
  - b-splines
- Surfaces
  - splines
  - nurbs
  - surface subdivision

Drawing Curves



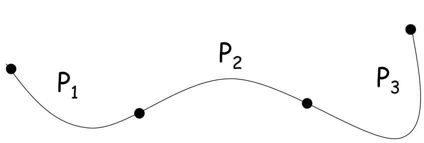
- Sample curve
- Draw line segments between sample points

Representing Curves

How should we represent a curve?

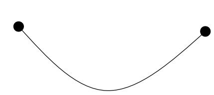
- Flexibility: Can we use the method for a wide range of curves?
- Efficiency: Can we sample it efficiently?
- Usability: Can a user specify it easily?

Complicated Curves



Simple curves connected end-to-end

Simple Curves



How should we represent a simple curve?

- Flexibility
- Efficiency
- Usability
- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

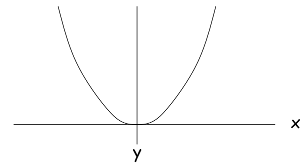
Curve Representation

- Explicit
- Implicit
- Parametric

Explicit

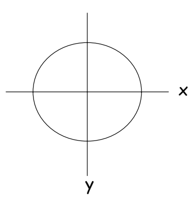
Curve is the trace of a function

Example: $y=x^2/4$



Explicit: flexibility

Many useful curves cannot be represented by explicit functions



Curve Representation

- ~~Explicit~~
- Implicit
- Parametric

Implicit

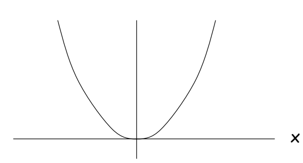
Curve is the zero loci of a function

Example: $f(x,y) = 4y-x^2$

Implicit

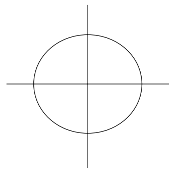
Curve is the zero loci of a function

Example: $f(x,y) = 4y-x^2$



Implicit: more flexibility

$F(x,y)=x^2+y^2-r^2$



But how could we describe a half circle?

Implicit: Efficiency

How can we find the zero loci of an function f(x,y)?

Curve Representation

- ~~Explicit~~
- ~~Implicit~~
- Parametric

Parametric

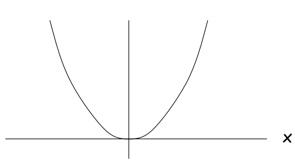
Curve is the range of a function

Example: $x = 2t$, $y = t^2$

Parametric

Curve is the range of a function

Example: $x = 2t$, $y = t^2$



Parametric curves

demo

Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
- Usability: not intuitive

Curve Representation

- ~~Explicit~~
- ~~Implicit~~
- ~~Parametric~~

OK -- I give up!

Curve Representation

- ~~Explicit~~
- ~~Implicit~~
- ~~Parametric~~

~~OK -- I give up!~~

Parametric: tradeoffs

- Flexibility: very expressive, easy to specify portions of curves
- Efficiency: easy to find points on curve
- Boundary conditions: easy to specify
- Usability: not intuitive without modeling tools!

Parametric cubic polynomials

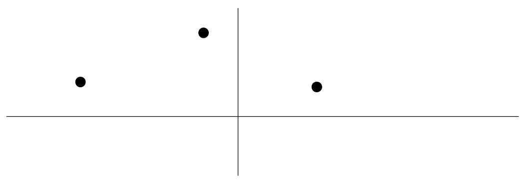
- Polynomials are expressive and can be efficiently computed
- Lower degree polynomials can't express non-planar curves
- Higher degree polynomials
  - Wiggle
  - Computationally more expensive

Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom
- Bezier
- B-spline

Interpolating polynomials

Give me a lowest degree polynomial curve through these points:



Interpolation

- Points: (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2)
- Compute: Quadratic polynomials $x(t)$, $y(t)$, $z(t)$ such that

  $(x(i),y(i),z(i)) = (x_i,y_i,z_i)$ for $i=0,1,2$

Exercise

Give me a parametric quadratic curve through the points

(1,0,2), (0,1,1), (2,-1,3)

Let's do x(t) together!

Computing x(t)

We want a quadratic function x(t) such that

$x(0)=1$, $x(1)=0$, $x(2)=2$

Step 1

Give me a quadratic polynomial x(t) such that:

- $x(1) = 0$
- $x(2) = 0$
- otherwise I don't care

Step 1

Give me a quadratic polynomial x(t) such that:

- $x(1) = 0$
- $x(2) = 0$
- otherwise I don't care

$x(t)=(t-1)(t-2)$

Step 2

Give me a quadratic polynomial x(t) such that:

- x(1) = 0
- x(2) = 0
- x(0) = 1

Step 2

Give me a quadratic polynomial x(t) such that:

- x(1) = 0
- x(2) = 0
- x(0) = 1

x(t)=(t-1)(t-2)/2

Step 3

Give me a quadratic polynomial x(t) such that:

- x(0) = 0
- x(1) = 0
- x(2) = 0

Step 3

Give me a quadratic polynomial x(t) such that:

- x(0) = 0
- x(1) = 0
- x(2) = 0

x(t)=0

Step 4

Give me a quadratic polynomial x(t) such that:

- x(0) = 0
- x(1) = 0
- x(2) = 2

Step 4

Give me a quadratic polynomial x(t) such that:

- x(0) = 0
- x(1) = 0
- x(2) = 2

x(t)=t(t-1)

Step 5

Give me a quadratic polynomial $x(t)$ such that:

- $x(0) = 1$
- $x(1) = 0$
- $x(2) = 2$

Step 5

Give me a quadratic polynomial $x(t)$ such that:

- $x(0) = 1$
- $x(1) = 0$
- $x(2) = 2$

$$x(t)=(t-1)(t-2)/2 + t(t-1) = (3/2)t^2 - (5/2)t + 1$$

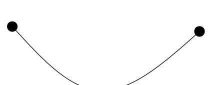
Exercise

You do $y(t)$ and $z(t)$! Write your results on the board.

General solution

$$x(t) = \Sigma_{i=0\ldots n-1} \frac{x_i\, [\Pi_{j=0\ldots n-1,\, j\neq i}\,(t-j)]}{[\Pi_{j=0\ldots n-1,\, j\neq i}\,(i-j)]}$$

Parametric Curves



How should we represent a simple curve?

- Flexibility
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- Boundary constraints: Can we specify continuity (including derivatives) at boundaries?

Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom
- Bezier
- B-spline

Hermitian splines

- Specify endpoint position
- Specify derivative at endpoint

Hermitian

- $X(t) = at^3 + bt^2 + ct + d$

- $X(0) = 3, \; X(1) = 2$
- $X'(0) = 1, \; X'(1) = 0$

- Write 4 equations that determine the coefficients a, b, c, and d.

Hermitian: Constraints

- X(0):
- X(1):
- X'(0):
- X'(1):

Hermitian: Constraints

- X(0): $d = 3$
- X(1): $a+b+c+d = 2$
- X'(0): $c = 1$
- X'(1): $3a+2b+c=0$

Hermitian Matrix Form

$$\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$$

Hermitian Matrix Form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

find X(t) for this example

Hint

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} = ?$$

X(t)

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 1 \\ 3 \end{pmatrix}$$

$X(t) = 3t^3 - 5t^2 + t + 3$

Verify

$X(t) = 3t^3 - 5t^2 + t + 3$

- $X(0) = 3,\ X(1) = 2$
- $X'(0) = 1,\ X'(1) = 0$

General Solution: X(t)

$x(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{pmatrix}$$

General Solution: Y(t)

$Y(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(1) \\ Y'(0) \\ Y'(1) \end{pmatrix}$$

General Solution: Z(t)

$Z(t) = at^3 + bt^2 + ct + d$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Z(0) \\ Z(1) \\ Z'(0) \\ Z'(1) \end{pmatrix}$$

Hermitian Basis Matrix

$$\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $Y'(0) = 0$, $Y'(1) = 1$

- Write the equations
- Plot the curve for t in [0,1]

Equations

- $X(0) = 3$, $X(1) = 2$, $X'(0) = 1$, $X'(1) = 0$
- $Y(0) = 2$, $Y(1) = 2$, $X'(0) = 0$, $X'(1) = 1$

- Equations:
- $X(t) = 3t^3 - 5t^2 + t + 3$
- $Y(t) = t^3 - t^2 + 2$

Plot

demo plot + reverse engineer

Hermitian Description

- Basis Matrix
- Basis (blending) Functions

Hermitian: X(t)

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

General Matrix: X

$$X(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Blending Functions: X

$$X(t) = \begin{bmatrix} P_1(t) & P_2(t) & P_3(t) & P_4(t) \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X'(0) \\ X'(1) \end{bmatrix}$$

Hermitian Blending Functions

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2t^3-3t^2+1, & -2t^3+3t^2, & t^3-2t^2+t, & t^3-t^2 \end{bmatrix}$$

$P_1(t)$ $P_2(t)$ $P_3(t)$ $P_4(t)$

Plot

demo

Hermitian: problem 1

Specifying derivatives is awkward,

particularly when many curves are connected with derivative continuity.

Parametric Continuity

C^i: The 0th, 1st, 2nd, …, ith derivative of adjacent curves agree at their boundary points.

Geometric Continuity

$G^0 = C^0$

For i>0, G^i means

- G^0 continuity plus
- The 1st, 2nd, …, ith derivative of adjacent curves are proportional at boundary point.

continuity

demo

Exercise

- First Hermitian curve x(t):
  - $x_1(0)$, $x_1(1)$, $x'_1(0)$, $x'_1(1)$
- Second Hermitian curve:
  - What conditions provide C^i & G^i continuity for i=0,1?

Hermitian: G^0

$x_2(0)=x_1(1)$

Hermitian: G^1

G^1: $x_2(0)=x_1(1)$ and $x_2'(0)=\alpha x_1'(1)$ for some α

(note: same α factor applies to y(t) and z(t))

Hermitian: C^0

$x_2(0)=x_1(1)$

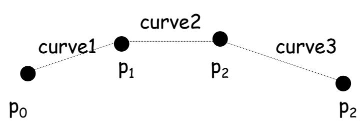
Hermitian: C^1

$x_2(0)=x_1(1)$ and $x_2'(0)=x_1'(1)$

Parametric cubic curves

- Interpolating
- Hermitian
- **Catmull-Rom** enforces C^1 continuity
- Bezier
- B-spline

Catmull-Rom Spline: C^1



tangent at $p_1 = (1/2) \langle p_2 - p_0 \rangle$

tangent at $p_2 = (1/2) \langle p_3 - p_1 \rangle$

Catmull-Rom Basis Matrix

- Compute the basis matrix for the Catmull-Rom spline from p_i to p_{i+1}

Catmull-Rom constraints

$X(t) = at^3 + bt^2 + ct + d$

assume $p_i=(x_i,y_i)$

- $X(0)=d=x_i$
- $X(1)=a+b+c=x_{i+1}$
- $X'(0)=c=(x_{i+1}-x_{i-1})/2$
- $X'(1)=3a+2b+c=(x_{i+2}-x_i)/2$

Catmull-Rom Basis Matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -.5 & 0 & .5 & 0 \\ 0 & -.5 & 0 & .5 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \end{bmatrix}$$

Catmull-Rom Basis Matrix

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \end{bmatrix}$$

Hermitian/Catmull-Rom: problem

Not invariant under affine transformations

In other words: transforming (rotate, scale, translate) the control points does not yield the transformed curve

Parametric cubic curves

- Interpolating
- Hermitian
- Catmull-Rom
- Bezier
- B-spline

Properties of Cubic Bezier Curves

- Control points p_0, p_1, p_2, p_3
- Curve starts at p_0 and ends at p_3.
- Line segments p_o-p_1 and p_3-p_2 are tangent to the curve at, respectively, p_0 and p_3.
- The curve lies within the convex hull of the control points.
- Curve is invariant under affine transformations.

Exercise

- Download bezier.cpp from /cs/cs155/labs
- Compile and Play
  - Right click to move red point
  - Left click to select new red point
  - Type "a" to then right click (3 times) to add new control points
  - Type "d" to delete last point